Appendix

A Comparison with Saddlepoint Approximation

We compare the Fourier transform based algorithm with a Saddlepoint Approximation method. For the Fourier transform based algorithm, we use a grid size of 2^{16} just as in the main text, which implies roughly the same computational time as in the main text. Since the Saddlepoint Approximation method provides an analytical solution for an approximation, the computational time is essentially zero. Therefore, instead of comparing computational time, we compare the precisions of the Fourier transform based algorithm and the Saddlepoint Approximation. Note that we could let the computational error of the Fourier transform based algorithm go to zero by letting the grid size go to infinity.

We will use an example similar to the one in the main text, $\mu = 0$, $\Sigma = I$, $\Gamma = 2I$, $\delta = 0$, $c = 0$, however, we will vary the number of factors: we will use $k = 6$, $k = 10$, and $k = 20$. An advantage of this specification is that there is a closed form solution for this special case, so that we can compute the exact value of the target capital. The reason for this is that this specification implies a χ^2 distribution with k degrees of freedom, since $Y = \frac{1}{2}X'\Gamma X + \delta' X + c = \sum_{i=1}^{k} X_i^2$ where $X_i \sim N(0,1)$ for $i = 1, ..., k$.

A.1 Closed Form Solution

In the following we denote by $\underline{\gamma}(s,z)$ and $\overline{\gamma}(s,z)$ the lower and upper incomplete gamma functions and by $\gamma(y)$ the complete gamma function.

The density for this specification is

$$f_k(y) = \frac{1}{2\gamma(k/2)} \left(\frac{y}{2}\right)^{k/2-1} \mathrm{e}^{-y/2} \qquad \text{for } y > 0\,,$$

and zero for $y \leq 0$. (Below this will not be mentioned separately.)

The cumulative distribution function is

$$F_k(y) = \frac{1}{\gamma(k/2)}\underline{\gamma}\left(\frac{k}{2}, \frac{y}{2}\right) = 1 - \frac{1}{\gamma(k/2)}\overline{\gamma}\left(\frac{k}{2}, \frac{y}{2}\right) .$$

We also have

$$J_k(y) = \int_0^y u f(u,k)\,\mathrm{d}u = kF_k(y) - 2y f_k(y)\,.$$

Note that this function has the properties $J_k(0) = 0$ and $J_k(\infty) = k$.

Define $y_0 = y_0(\epsilon, k)$ by

$$F_k(y_0) = \epsilon\,.$$

With this one obtains the target capital $\mathrm{TC}_k(\epsilon)$ (up to the sign, which is a matter of convention)

$$\mathrm{TC}_k(\epsilon) = \frac{1}{\epsilon} J_k(y_0) = k - \frac{2y_0}{\epsilon} f_k(y_0)\,.$$

A.2 Saddlepoint Approximation

In the following, we provide a brief description of the Saddlepoint Approximation. For more details and an overview of the literature, see Broda and Paolella (2011, p. 83ff)

First, consider the general case for arbitrary distributions. Take a density that can be written in the following way for some function $\varphi(\cdot)$:

$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(t)\,\mathrm{e}^{-ity}\mathrm{d}t\,.$$

Consider a change of variables in the integral with a new integration variable $t = -is$ and denote $M(s) = \varphi(-is)$.

Then using

$$\varphi(t) = \int_{-\infty}^{\infty} f(y)\,\mathrm{e}^{ity}\mathrm{d}y\,,$$

one obtains

$$M(s) = \varphi(-is) = \int_{-\infty}^{\infty} f(y)\, \mathrm{e}^{sy} \mathrm{d}y \,.$$

Note that M is the moment generating function, i.e., one has $M(0) = 1$, and the mth derivative at $s = 0$ gives the corresponding momentum, $M^{(n)}(0) = m_n = \int y^n f(y)\, \mathrm{d}y$.[6]

Introducing $K(s) = \log(M(s))$ (the cumulant generating function), we can write f as

$$f(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp\left(K(s) - sy\right)\, \mathrm{d}s \,.$$

The saddlepoint is given by solving the equation $K'(s) = y$ yielding $\hat{s}(y)$. Using this, the Saddlepoint Approximation (SPA) of $f(y)$ is

$$f^{\mathrm{SPA}}(y) = \frac{1}{\sqrt{2\pi K''(\hat{s}(y))}} \exp\left(K(\hat{s}(y)) - y\hat{s}(y)\right) \,.$$

For the cumulative distribution function one has

$$F^{\mathrm{SPA}}(y) = \Phi_{\mathrm{SN}}(\hat{w}) + \phi_{\mathrm{SN}}(\hat{w}) \left(\frac{1}{\hat{w}} - \frac{1}{\hat{u}}\right)$$

where $\hat{w} = \mathrm{sign}(\hat{s})\sqrt{2(y\hat{s} - K(\hat{s}))}$, $\hat{u} = \hat{s}\sqrt{K''(\hat{s})}$. Φ_{SN} and ϕ_{SN} refer the cumulative distribution function and probability density function of the standard normal distribution, respectively.

The integral $J(y)$ in the expected shortfall in this approximation is

$$J^{\mathrm{SPA}}(y) = \mu\Phi_{\mathrm{SN}}(\hat{w}) + \phi_{\mathrm{SN}}(\hat{w}) \left(\frac{\mu}{\hat{w}} - \frac{y}{\hat{u}}\right) \approx J(y) = \int_{-\infty}^{y} u f(u)\, \mathrm{d}u \,,$$

where $\mu = \mathrm{E}(Y)$. See Lugannani and Rice (1980) for the details how f^{SPA}, F^{SPA}, and J^{SPA} can be obtained.

Note that for the χ^2 distribution the lower tail decays faster than by a Gaussian (it extends until a finite distance from the peak), while the upper tail decays slower, roughly as $\exp(-y/2)$ instead of $\exp(-y^2/2)$.

[6] Note that $M(s)$ is real.

A.3 SPA for the χ^2 distribution

We now consider SPA for the special case of a χ^2 distribution. One has

$$\varphi(t) = \int_y^\infty f_k(y)\, \mathrm{e}^{ity} \mathrm{d}y = (1-2it)^{-k/2}\,, \qquad y > 0\,,$$

and for $1 - 2s > 0$

$$M(s) = \varphi(-is) = (1-2s)^{-k/2}\,,$$

This gives the known values $\mu = k$, $\mathrm{Var}(Y) = 2k$. Further, one obtains

$$\begin{aligned} K(s) &= \log(M(s)) = -\frac{1}{2} k \log(1-2s)\,, \\ K'(s) &= \frac{k}{1-2s}\,, \\ K''(s) &= \frac{2k}{(1-2s)^2}\,. \end{aligned}$$

The solution of the saddlepoint equation $K'(s) = y$ is

$$\hat{s}(y) = \frac{y-k}{2y}$$

(Note that for $y > 0$ this indeed satisfies the condition $\hat{s}(y) < 1/2$.)

$$K(\hat{s}(y)) - y\hat{s}(y) = -\frac{1}{2}\left(y - k - k\log(y/k)\right)$$

One has

$$\hat{w}(y) = \mathrm{sign}(y-k)\sqrt{y - k - k\log(y/k)}$$

and

$$\hat{u}(y) = \frac{y-k}{\sqrt{2k}}$$

One obtains

$$f_k^{\mathrm{SPA}}(y) = \frac{1}{y}\,\frac{k}{4\pi}\exp\left[-\frac{1}{2}\left(y - k - k\log(y/k)\right)\right]$$

A.4 Comparison of the Saddlepoint Approximation and the Fourier Transform Based Algorithm

Table 1 compares the exact value of the target capital that can be computed analytically for this special case, the values of the TC computed using SPA, and the TC computed using the Fourier transform based algorithm.

We report values for $\epsilon = 0.01$ and $k = 6$, $k = 10$, and $k = 20$. The table shows that the relative errors for both the SPA and the Fourier transform based algorithm decreases as k increases. However, the relative error for the Fourier transform based algorithm is much lower (less than 10^{-8}) than for SPA (between 1.5% and 24%).

k	TC	TC_{SPA}	relative error	$TC_{Fourier}$	relative error
6	0.63929	0.51356	0.2448	0.63929	5.733×10^{-9}
10	2.0596	1.9366	0.0635	2.0596	4.377×10^{-14}
20	7.1987	7.0943	0.0147	7.1987	5.552×10^{-15}

Table 1: Comparison of the exact value of the target capital (TC), the Saddlepoint Approximation (TC_{SPA}), and the Fourier transform based algorithm ($TC_{Fourier}$). The fourth and the sixth column report the computational error compared to the exact value. Observe that the difference between the exact value and the Fourier transform based value is not visible when comparing the second and the fifth column since we report only five digits in the table, whereas the error is in the 9th, 14th, and 15th digit, respectively. The Fourier transform based algorithm uses a grid size of 10^{16}.

Note that we have provided a convex function $y(x) = \sum_j y_j(x)$ (which corresponds to a positive definite matrix Γ). Convexity seems plausible, for at least two reasons. First, the Swiss Solvency Test specifies a convex effect of the interest rate shock on asset values. Second, hedging by insurance companies typically involves convex instruments that reduce "tail risk", e.g. by buying put options on underyling equities or convertible bonds. For such positions the risk bearing capital is a convex function in the value of the underlying share price since for low prices of the underlying these positions are less sensitive to price changes than for high prices of the underlying. Our numerical calculations suggest that for convex functions $y(x)$ (in the provided example $\Gamma = I$), the SPA is not precise due to the steep fall of the function $y(x)$ at lower values.

However, it should be noted that our calculations for $\Gamma = -2I$ (that is a concave $y(x)$) suggest that for concave functions $y(x)$ the SPA is doing better than for convex functions: e.g. for $k = 6$ the approximation error of SPA is only 2%.