B Uniqueness of Equilibrium

Our analysis has focused on a particular equilibrium, namely the full-trade one. It can be easily seen that in the full-trade class, the equilibrium is unique: equations (15) and (16) uniquely determine the spread θ, and the marginal types $\underline{v}$ and $\overline{c}$ are then uniquely determined in market with balanced entry. Outside the full-trade class, there are multiple equilibria in general. First, there is a trivial no-trade equilibrium with no entry on both sides.

Second, if one side (say, sellers) has all the bargaining power, $\alpha_B = 0$, then the optimal mechanism involves no participation fees on the other side, $K_B^* = 0$. At least in the static setup, $\delta = 0$, this leads to the existence of multiple equilibria in which the market is imbalanced: the mass of entering buyers is larger than the optimal one, while the mass of sellers is smaller. In these equilibria, the marginal buyer's type $\tilde{v} < \underline{v}$, while the marginal seller's type $\tilde{c} = \tilde{v} - K_S^*$. The entering sellers are matched with probability 1 and offer the price $\tilde{v}$, which is accepted. The entering buyers are matched with probability $\frac{F_S(\tilde{c})}{1-F_B(\tilde{v})} < 1$.[29]

The multiplicity of equilibria identified above hinges on $K_B^* = 0$ so that the buyers with $v < \underline{v}$ may enter the market even though they are indifferent. The next proposition shows that the equilibrium is *unique* if both sides have some bargaining power, $\alpha_B, \alpha_S > 0$, which, according to Proposition 2, leads to positive fees on both sides, $K_B^*, K_S^* > 0$.

Proposition 4. *If $\alpha_B, \alpha_S > 0$, the profit maximizing balanced full trade equilibrium is the unique non-trivial equilibrium in the static setup ($\delta = 0$) given the profit maximizing fees K_B^* and K_S^*.*

The proof is by contradiction. We show that if there were either more or less entry than optimal on either side of the market, then the marginal trader types would not be able to recover their participation costs and therefore would rather prefer to stay out.

Proof of Proposition 4. We argue by contradiction. Suppose there is a different equilibrium with marginal buyer and seller types $\tilde{v} \neq \underline{v}$ and $\tilde{c} \neq \overline{c}$. We restrict attention to equilibria with trade, so $\tilde{v} < 1$ and $\tilde{c} > 0$.

Step 1. We first show that the equilibrium cannot involve an entry expansion on either side: $\tilde{v} < \underline{v}$ or $\tilde{c} > \overline{c}$. We only prove the result for the buyers; the proof for the sellers is parallel. Consider the (gross of the participation cost) profit for a $\tilde{v}$-type buyer who offers price p_B in

[29] To show that this is in fact an equilibrium, we also need to check that the $\tilde{c}$-type seller will not have an incentive to deviate to a price higher than $\tilde{v}$. This is true at least if $\tilde{v}$ is sufficiently close (from below) to the full-trade marginal type $\underline{v}$. Indeed, the marginal profit for $\tilde{v}$-type buyer is proportionate to $\tilde{c} - J_B(\tilde{v})$, which by continuity is close to $\underline{c} - J_B(\underline{v}) = -\frac{F_S(\overline{c})}{f_S(\overline{c})} < 0$. If the distribution $F_B(\cdot)$ has a decreasing hazard rate, then it can be shown that such an equilibrium exists for any $\tilde{v} < \underline{v}$.

equilibrium:

$$\hat{\pi}_B(p_B) = \alpha_B \min\left\{1, \frac{F_S(\tilde{c})}{1-F_B(\tilde{v})}\right\}\frac{F_S(p_B)}{F_S(\tilde{c})}(\tilde{v}-p_B)$$

where the min term is the probability that the buyer is matched. We now show that the probability of matching is bounded from above as

$$\begin{aligned}\min\left\{1, \frac{F_S(\tilde{c})}{1-F_B(\tilde{v})}\right\} &= F_S(\tilde{c})\min\left\{\frac{1}{F_S(\tilde{c})}, \frac{1}{1-F_B(\tilde{v})}\right\}\\ &\leq F_S(\tilde{c})\frac{1}{1-F_B(\tilde{v})} < F_S(\tilde{c})\frac{1}{1-F_B(\underline{v})}\\ &= F_S(\tilde{c})\frac{1}{F_S(\overline{c})}\end{aligned}$$

where the second inequality follows from our assumption $\tilde{v} < \underline{v}$, and the third from the fact that the market is balanced in the full-trade equilibrium, $1-F_B(\underline{v}) = F_S(\overline{c})$. Substituting this bound into $\hat{\pi}_B(p_B)$ and cancelling out $F_S(\tilde{c})$, we obtain

$$\hat{\pi}_B(p_B) < \alpha_B\frac{F_S(p_B)}{F_S(\overline{c})}(\tilde{v}-p_B) < \pi_B^*,$$

where π_B^* is the expected (gross) equilibrium profit of the $\underline{v}$-type buyer in the original full-trade equilibrium. Since the $\underline{v}$-type buyer breaks even in the full-trade equilibrium, $\pi_B^* \leq K_B^*$. It follows that the type-$\tilde{v}$ buyer will not recover the participation cost: $\hat{\pi}_B(p_B) < K_B^*$. This is a contradiction.

Step 2 Contraction on both sides of the market cannot be an equilibrium. That is, we cannot have $\tilde{c} < \bar{c}$ and $\tilde{v} > \underline{v}$. We will show that the marginal utility of increasing the price is positive for the marginal buyer in any equilibrium with two sided contraction. Since the same argument holds also for sellers, we know that buyers price at $\tilde{c}$ and sellers at $\tilde{v}$. Then we will show that an equilibrium with this pricing cannot exist for a spread larger than the profit maximizing one, i.e. $\tilde{v} - \tilde{c} > \underline{v} - \bar{c}$.

The marginal $\tilde{v}$-type buyer's utility when setting price p_B is

$$\hat{\pi}_B(p_B) = \alpha_B(\tilde{v}-p_B)\frac{F_S(p_B)}{F_S(\tilde{c})}\frac{\min\{B,S\}}{B}$$

The marginal profit $\hat{\pi}'_B(p_B)$ of the marginal buyer has the same sign as

$$\begin{aligned}-F_S(\tilde{c}) + (\tilde{v}-\tilde{c})f_S(\tilde{c}) &= (\tilde{v} - J_S(\tilde{c}))f_S(\tilde{c})\\ &> (\underline{v} - J_S(\tilde{c}))f_S(\tilde{c})\\ &\geq (\underline{v} - J_S(\bar{c}))f_S(\tilde{c}) \geq 0\end{aligned}$$

where the first two inequalities follow from $\tilde{v} > \underline{v}$ and $\tilde{c} < \bar{c}$ and the third from the fact that there is full trade at $\underline{v}$, $\bar{c}$. A positive $\hat{\pi}'_B$ means that the buyer will set a price equal to the cost of the marginal seller $\tilde{c}$. By monotonicity, all other buyers (who have $v > \tilde{v}$), will also price at $\tilde{c}$. By an analogous argument, all sellers will price at $\tilde{v}$. Since the probability of trading conditional on being matched is 1, the utility of the marginal types is

$$\alpha_B(\tilde{v} - \tilde{c})\frac{\min\{B, S\}}{B} = K_B^* \tag{40}$$

$$\alpha_S(\tilde{v} - \tilde{c})\frac{\min\{B, S\}}{S} = K_S^* \tag{41}$$

Dividing the two equations and using $\alpha_B/\alpha_S = K_B^*/K_S^*$ we get $B/S = 1$. Substituting this back into (40) and (41) gives us

$$\alpha_B(\underline{v} - \bar{c}) < K_B^*, \qquad \alpha_S(\underline{v} - \bar{c}) < K_S^*,$$

since $\tilde{v} - \tilde{c} > \underline{v} - \bar{c}$. This is a contradiction to the marginal conditions (15) and (16) for the full-trade equilibrium. Hence, a contraction equilibrium cannot exist. □