For-Profit Search Platforms*

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Abstract

We consider optimal pricing by a profit-maximizing platform running a dynamic search and matching market. Buyers and sellers enter in cohorts over time, meet and bargain under private information. The optimal centralized mechanism, which involves posting a bid-ask spread, can be decentralized through participation fees charged by the intermediary to both sides. The sum of buyers' and sellers' fees equals the sum of inverse hazard rates of the marginal types and their ratio equals the ratio of buyers' and sellers' bargaining weights. We also show that a monopolistic intermediary in a search market may be welfare enhancing.

Keywords: Dynamic random matching, two-sided private information, intermediaries

JEL Codes: D82, D83

1 Introduction

Search market intermediaries that charge participation fees to traders and let them search for and bargain with trading partners play an important role in many markets. Examples include online trade and auction web sites, job search platforms, online freelancer and tutoring platforms, and markets created by credit card issuers. Such search market intermediaries (or platforms) have been at the center of attention for quite some time and have received increased attention recently, in areas as diverse as labor markets, competition policy, and international

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trade and development. This attention seems justified by the money at stake: as an example, intermediation services account for over a quarter of GDP in the US (Spulber, 1999, p. 21).

Search platforms differ in who pays the fee. Job search platforms that target a general audience typically let employers (that is, buyers) pay a job posting fee and make participation for job seekers free.[1] In these job markets, the employers typically have the stronger bargaining position, especially for low-skill labor. Contrary to this, in high-skill, specialized labor markets employees have a stronger bargaining position. An example of a platform catering to such high-skill "\$100k" job markets, TheLadders.com, charges job seekers \$180 per year additionally to the job posting fees paid by employers.[2]

In a different kind of job market – the market for tutors – the tutor typically offers the student the price for his or her services. An example of a platform for tutors, Tutorz.com, charges a subscription fee to tutors and makes participation free for students.[3] This pattern – that the seller sets the price and pays fees to be able to participate on the platform, whereas participation for buyers is free – is common in many other markets. Examples include malls and convention centers.

What the above-mentioned examples of platforms have in common is their decentralized nature. They do not dictate prices or store the goods. In other words, they do not make markets in the variety of goods or services they offer. Instead, they offer services to buyers and sellers, such as facilitating search or matching. Rather than through centrally setting prices, they raise their revenues by charging fees to buyers and sellers.

There are several reasons why centralization may be too costly or not feasible. One reason is that transactions may not be observable (or costly to observe). As an example, a job search platform may not know whether a candidate eventually took the job or not. A buyer and seller (or employer and employee) who found each other through the platform may collude to withdraw from the platform without officially trading and later trade bilaterally, thereby avoiding payments to the platform. Another argument for decentralization is that the platform

[1] For example, Monster.com charges employers \$395 per job posting and Careerbuilder.com charges \$419 per job posting. Participation is free for job seekers on both platforms (see the websites Monster.com and Careerbuilder.com, accessed on April 10, 2013).

[2] In the past, TheLadders.com made participation free for employers (see Caplan (2008)). Currently, both job seekers and employers pay fees (see TheLadders.com, website accessed on April 10, 2013).

[3] Tutorz.com is currently transitioning from a per month subscription fee to a per match fee (see www.tutorz.com/student/pricing, accessed on April 10, 2013). As we show later, per match fees and subscription fees are equivalent in our setup.

may lack information needed to implement the optimal bid and ask prices.[4] In particular, a misjudgement of demand or supply may result in excess inventories, with associated logistic costs that may be prohibitively high. A decentralized market is less prone to such problems. Additionally, decentralization may also alleviate the need for third-degree price discrimination.[5]

The structure of platforms' fees is at the core of their business strategy. The above examples suggest the following pattern: if the price is set predominantly by one side (for example, employers, tutors, or sellers), then this side pays the participation fee to the platform. More generally, the side with a stronger bargaining power pays a higher fee.

Motivated by these examples, we develop a theory of optimal pricing by profit-maximizing platforms in a search market. Our model is stylized to capture some of the main aspects of such platforms. First, traders have private information about their valuation for the good *before* they decide to join the platform. Second, we consider *dynamics* in such markets. Hence buyers and sellers have an option value of future trade and distributions in the market are endogenous, since inefficient traders may need a longer time to trade and may be overrepresented in the market. Third, prices are formed through bargaining among buyers and sellers.

In our model, buyers and sellers arrive to the intermediary's platform over time, match and bargain. Sellers have one unit of an indivisible homogeneous good, buyers have unit demand. The matching technology is constant returns to scale. The bargaining protocol is assumed to be a random proposer one: a buyer makes a take-it-or-leave-it offer with some probability, and the seller makes such an offer with a complementary probability. The proposal probability is a stylized way of modelling the bargaining power in a private information setup. Two important special cases are that the seller posts a price (that is, the seller gets to propose with probability 1) and that the buyer posts a price. This constitutes what we call a dynamic matching and bargaining game.

We focus on the following questions. What is the profit-maximizing allocation rule? Can this allocation rule be implemented by simple participation fees? What is the optimal fee structure: should the buyer, the seller, or both pay the fee? How is it related to the bargaining power of the two sides of the market?

[4]Hayek (1945) was first to argue for the benefits of decentralization when information is dispersed among many agents.

[5]Assume, for example, that a job search platform cannot price discriminate between jobs for engineers and jobs for janitors. Centrally prescribing the same wage for an engineer and a janitor would be inefficient, whereas charging the same fee for posting an engineering and a janitorial position may be a reasonable approximation of the optimal pricing.

As a benchmark, we first analyze a setup in which the intermediary can centralize the exchange. We show that the optimal centralized mechanism involves buying the good from the sellers at a bid price, selling it to the buyers at an ask price, and collecting the spread between the two prices as the revenue. The optimal spread balances the market in each period in the sense that the mass of entering buyers equals the mass of entering sellers. Further, the spread is equal to the sum of the buyer's and the seller's inverse hazard rates at the margin (which can be seen as the semi-elasticities of demand and supply at the margin). In this optimal mechanism, the most efficient traders, that is, buyers with valuations above the ask price, and sellers with costs below the bid price, enter and trade immediately upon matching. All other traders do not enter. We call this the full-trade property.

The maximal revenue from the optimal mechanisms provides an upper bound on the intermediary's profit in a bargaining market. Our main result is that this upper bound is attainable by setting appropriate participation fees and letting buyers and sellers meet and bargain in a decentralized market. Therefore, the platform can achieve the benchmark centralized profit in a decentralized market with participation fees. With optimal fees, the decentralized market has a full-trade equilibrium: all sellers propose prices equal to the marginal participating buyer's valuation, and all buyers propose prices equal to the marginal participating seller's cost. These prices are accepted with probability 1.

Our general results on the per period fee structure are the following. The sum of the buyer and seller fees is equal to the spread of the optimal mechanism and hence, as described previously, equal to the sum of the inverse hazard rates at the margin. The ratio of fees for the buyer and seller is equal to the ratio of the bargaining weights. In other words, the side with a greater bargaining power ends up paying a higher fee. In the extreme case in which say the seller posts a price (that is, the seller has all the bargaining power), the seller pays the entire fee and participation is free for buyers. This supports the pattern suggested by the real-world examples given above.[6]

The result that decentralization is possible is not self-evident since the intermediary only

[6] This result sheds further light on the issue of centralization versus decentralization. In a static variant of our model, assume aggregate uncertainty of a simple form: both valuations and costs (that is, the demand and supply) are subject to some additive random shift. Then in order to implement the optimal bid and ask prices, the intermediary would need to know this random shift. Such knowledge may not be always forthcoming. At the same time, the spread can be easily shown *not* to depend on the shift, and therefore the optimal participation (or per match) fees are also invariant to the shift. This implies that the profit-maximizing outcome can still be implemented in our decentralized matching and bargaining game.

has two choice variables (participation fees of buyers and sellers), but has to control at least four variables (the entry choices of buyers and sellers and the prices set by buyers and sellers). The basic intuition for this result can be seen in a static setup in which the seller always sets the price. The intermediary wants to make sure that the price set by the marginal seller is not higher than the valuation of the marginal buyer to ensure full trade. This reduces to the requirement that the marginal seller's optimal markup is not larger than the intermediary's optimal bid-ask spread. The intuition for why this requirement is fulfilled is that the intermediary wants less trade than the marginal seller, since he has to give rents to both sides and not just one. Therefore, the intermediary's optimal spread is larger than the marginal seller's optimal markup.

In most of the paper, we consider participation fees that have to be paid in each period a trader is in the market. Two alternative ways of raising revenues is through per match fees and transaction fees. Per match fees are only paid if a trader is matched, transaction fees are only paid if a transaction occurs. In our setup, the optimal mechanism can be implemented with participation fees and per match fees, but not with transaction fees. The reason for the optimality of per match fees is that they are equivalent to participation fees in a balanced market in which a trader is matched in every period with probability 1. The reason transaction fees are different is that in contrast to participation and per match fees they are not sunk at the moment of bargaining. To see why it matters that transaction fees are not sunk, consider a static setup. Recall that in the optimal mechanism, everyone who enters should trade with probability 1. However, a seller with high costs is willing to enter even if his offer is rejected with a high probability, since in case the offer is rejected, he does not have to pay transaction fees. In case his offer is accepted, he makes a positive profit. With participation (or per match) fees, a seller is not willing to make such a gamble: if his offer is rejected, he still has to pay the participation fees and hence incurs a loss.

Literature This paper relates to three strands of literature: dynamic random matching (see, for example, Rubinstein and Wolinsky (1985), Gale (1987), Wolinsky (1988), Mortensen and Wright (2002), Satterthwaite and Shneyerov (2007, 2008), Atakan (2007a,b), Shneyerov and Wong (2010a,b), Lauermann (forthcoming), Lauermann, Merzyn, and Virág (2011)), intermediaries (see, for example, Rubinstein and Wolinsky (1987), Gehrig (1993), Spulber (1996), Rust and Hall (2003), Loertscher and Niedermayer (2012)), and two-sided markets (see, for example, Caillaud and Jullien (2003), Armstrong (2006), Rochet and Tirole (2006), Nocke, Peitz, and Stahl (2007), Goos, Van Cayseele, and Willekens (2011)).

We depart from the dynamic random matching literature by assuming that the search and matching platform is owned by a profit-maximizing intermediary. A full-trade equilibrium, which is at the core of our analysis, has been previously identified in Satterthwaite and Shneyerov (2007) and, in a model with pairwise matching and bargaining as here, in Shneyerov and Wong (2010a). It turns out that if the search costs incurred by traders are endogenously determined, as fees charged by the intermediary, the equilibrium has the full-trade property. This enables us to obtain our characterization of the optimal fee structure and a simple, tractable equilibrium characterization that is a helpful starting point for further analysis.

Our contribution to the intermediation literature is that we let the intermediary design a mechanism that takes into account the possibility of delay, of changing the steady state distributions in the market, and traders' option values of future trade, and lets buyers and sellers meet randomly and bargain bilaterally. In particular, the monopolistic intermediating platform in this paper differs from the middlemen in Rubinstein and Wolinsky (1987), Rust and Hall (2003), Loertscher and Niedermayer (2012) that do not have monopoly power and take the market equilibrium as given. This difference – that the intermediary has monopoly power and can shape the distributions in the market – is at the core of the current paper. One of our main questions is how an intermediary chooses the exchange mechanism in a given period taking into account two channels through which this affects traders' behavior in other periods. First, in preceding periods, traders consider the option value of delaying trade. Second, traders that did not trade in the given period stay in the market and hence affect the distribution of types in subsequent periods. We further differ from Loertscher and Niedermayer (2012) by having a continuum of traders rather than one buyer and one seller (or a small number of buyers and sellers). Further, we provide a justification for the monopolistic market maker considered in Spulber (1996, Section 4) and Rust and Hall (2003, Corollary 5.2) by showing that setting bid-ask prices is indeed an optimal mechanism.[7]

A direct comparison with the two-sided markets literature is not possible, since the two-sided markets literature typically deals with buyers and sellers having a reduced form utility function that depends directly on the number of participants on the other side of the market

[7]Section 4 in Spulber (1996) and Corollary 5.2 in Rust and Hall (2003) deal with a monopolistic market maker. Other parts of these articles deal with competing market makers and/or middlemen.

(see, for example, Armstrong (2006)).[8] Caillaud and Jullien (2003) and Goos, Van Cayseele, and Willekens (2011) do provide a random-matching microfoundation, but gains from trade are public information and always positive, whereas in our analysis the focus of attention is on the heterogeneity of agents' private types, private information bargaining, and the platform's strategy to avoid a breakdown of bargaining in equilibrium.[9] Rochet and Tirole (2006) mostly deal with reduced form utility functions, but also consider an extension with a random-matching microfoundation with two-sided private information (see Rochet and Tirole, 2006, pp. 661-663). In our setup agents know their type *ex ante*, that is, before choosing whether to join the platform, whereas in Rochet and Tirole (2006) agents find out their type *ex post*, that is, after joining the platform. This difference is crucial: in contrast to Rochet and Tirole (2006), in our model it is not possible to implement first-best by using fees collected ex ante to subsidize trade ex post. Further, our main difference in terms of predictions is that in our setup the optimal ratio of fees is equal to the ratio of bargaining weights rather than the ratio of the elasticities of demand and supply.

2 Setup

We consider an infinite horizon model of a market for a homogeneous, indivisible good. Time is discrete, $\tau = 0, 1, 2, \ldots$. In period $\tau = 0$ the market starts out empty. Each period, a unit mass of newborn buyers and sellers is available for entry in the market. There is a friction of discounting; the discount factor is denoted as δ. Each buyer has a unit demand for the good, while each seller has unit supply. All traders are risk neutral.

Potential buyers are heterogeneous in their valuations v of the good. Potential sellers are also heterogeneous in their costs c of providing the good. In the following, we will also refer to valuations and costs as types. The types of new potential buyers are private information and are drawn independently from the continuously differentiable cumulative distribution function $F_B(v)$ for buyers and $F_S(c)$ for sellers. Densities are denoted with $f_B(v)$ and $f_S(c)$ and have support $[0, 1]$. We make the standard assumption that Myerson's regularity condition holds,

[8] See Nocke, Peitz, and Stahl (2007) for a paper that has an imperfect-competition microfoundation for the reduced form utility functions.

[9] Additionally, Caillaud and Jullien (2003) assume that gains from trade are constant for all traders, so that we also differ in terms of the heterogeneity of traders. Goos, Van Cayseele, and Willekens (2011) additionally assume non-transferable utility in the main part of their paper, in which they analyze participation, per match, and transaction fees. In an extension, they consider transferable utility, but only participation fees.

that is, the virtual type functions

$$J_B(v) := v - \frac{1 - F_B(v)}{f_B(v)}, \qquad J_S(c) := c + \frac{F_S(c)}{f_S(c)},$$

are increasing. Each trader's type will not change once it is drawn. Entry (or participation, or being active) is voluntary. Potential traders decide whether to enter the market once they are born.[10] Those who do not enter will get zero payoff.

First, we will derive the optimal direct mechanism as a benchmark. We will show that it is optimal for the intermediary to choose an anonymous, stationary mechanism.

After deriving the optimal mechanism, we will consider the optimal dynamic random matching and bargaining with participation fees implementation in Section 4. Here, the intermediary could in principle choose non-anonymous, non-stationary fees. That is, the fee offered to a trader might depend on his identity and on the period in which the he entered the market. The endogenous distributions of the types of traders who have not traded yet might change over time. However, because the optimal mechanism is anonymous and stationary, it is sufficient to only consider anonymous, stationary fees.

We assume throughout the paper that the intermediary can fully commit.

3 Optimal Direct Mechanism

Consider a mechanism in which all buyers and sellers report their valuations and costs, respectively, and the intermediary can arbitrarily match buyers and sellers. Since the good is homogeneous, this can be expressed in the following equivalent way: the goods of sellers who sell enter a pool, buyers get their goods from this pool. The only constraint is that the same number of sellers sell as buyers buy. A buyer's or seller's probability of trade and transfers are determined by his or her report. The highest profits that can be achieved in such a centralized market will serve as an upper bound for the profits in the decentralized market, where the intermediary cannot influence how buyers and sellers are matched.

Index the mass 1 of buyers being born in each period τ with $i_0 \in [0, 1)$.[11] Each buyer in the continuum of buyers ever born is uniquely identified by the tuple (τ, i_0), where τ is the period

[10] One could alternatively think of potential traders also having the possibility to delay entry. This will not make a difference in our setup, since entry turns out to be either always or never profitable for a trader, so that in equilibrium no one would delay entry.

[11] i_0 can be thought of in the following way: order all buyers in some order (this order is random, since they are ex ante identical) and label the buyer who has mass i_0 buyers to his left as i_0.

a buyer is born. Similarly, sellers are indexed by $j_0 \in [0,1)$ in each period and are uniquely identified by (τ, j_0). To simplify notation, we will use the indices $i := \tau + i_0$ and $j := \tau + j_0$ to identify buyers and sellers rather than (τ, i_0) and (τ, j_0). We denote the birth periods of buyer i and seller j as $\tau(i) := \lfloor i \rfloor$ and $\tau(j) := \lfloor j \rfloor$, respectively.[12] A *direct mechanism* is defined as

$$\mathcal{M} := \left\{ q_{i,s}^B(v), q_{j,s}^S(c), t_{i,s}^B(v), t_{j,s}^S(c) : s = 0, 1, 2, ...;\ i, j \in \mathbb{R}_{0+};\ v, c \in [0,1] \right\},$$

where $q_{i,s}^B(v)$ is buyer i's probability of trade in period s if he reports type v and similarly $q_{j,s}^S(c)$ is seller j's probability of trade in period s if he reports c. Two natural restrictions are that buyers and sellers cannot trade before being born (that is, $q_{i,s}^B(v) = 0$ for all $s < \tau(i)$ and $q_{j,s}^S(c) = 0$ for all $s < \tau(j)$) and that the total probability of trade cannot exceed 1 (that is, $\sum_{s=0}^{\infty} q_{i,s}^B(v) \leq 1$ for all i, v and $\sum_{s=0}^{\infty} q_{j,s}^S(c) \leq 1$ for all j, c). Further, define $t_{i,s}^B(v)$ and $t_{j,s}^S(c)$ to be payments from buyer i and payments to seller j in period s, respectively. Note that it is not necessary to condition $q_{i,s}^B$, $q_{j,s}^S$, $t_{i,s}^B$, and $t_{j,s}^S$ on the reports of the other participants, because there is a continuum of traders and by the law of large numbers the realized distribution of types is common knowledge. Further, by the revelation principle, we restrict attention to truthful reporting.

We assume that in each period s, the mechanism must involve a balanced market: the quantity of goods bought is required to be equal to the quantity sold. This trade balance restriction takes the form

$$\int_0^{\infty} \int_0^1 q_{i,s}^B(v) dF_B(v) di = \int_0^{\infty} \int_0^1 q_{j,s}^S(c) dF_S(c) dj, \tag{1}$$

for each period $s = 0, 1, 2, \ldots$. Note that having an infinite upper bound for the integral with respect to i is not a problem, since we impose the constraint $q_{i,s}^B(v) = 0$ for $\tau(i) > s$, that is, a buyer cannot trade before being born. The same applies to sellers.

The intermediary's goal is to choose a mechanism that maximizes the discounted expected transfers from buyers and sellers, subject to incentive compatibility (IC), individual rationality (IR), and the trade balance constraints. Define the buyer's expected utility given his true type is v and his reported type $\hat{v}$,

$$U_i^B(v, \hat{v}) = \sum_{s=0}^{\infty} \delta^{s-\tau(i)} \left(v q_{i,s}^B(\hat{v}) - t_{i,s}^B(\hat{v}) \right),$$

[12] $\lfloor i \rfloor$ denotes the floor function applied to i, that is, the largest integer not greater than i.

and similarly for the seller,

$$U_j^S(c,\hat{c}) = \sum_{s=0}^{\infty} \delta^{s-\tau(j)} \left(t_{j,s}^S(\hat{c}) - c q_{j,s}^S(\hat{c}) \right).$$

The intermediary's objective is to choose $\mathcal{M}$ that will maximize

$$T := \sum_{s=0}^{\infty} \delta^s \int_0^{\infty} \int_0^1 t_{i,s}^B(v) dF_B(v) di - \sum_{s=0}^{\infty} \delta^s \int_0^{\infty} \int_0^1 t_{j,s}^S(v) dF_S(c) dj. \tag{2}$$

subject to

$$U_i^B(v,v) \geq U_i^B(v,\hat{v}), \qquad U_j^S(c,c) \geq U_j^S(c,\hat{c}), \tag{IC}$$
$$U_i^B(v,v) \geq 0, \qquad U_j^S(c,c) \geq 0, \tag{IR}$$

for all i, j, v, $\hat{v}$, c, $\hat{c}$, and the trade balance (1). Such an optimal mechanism is completely characterized by the following proposition.

Proposition 1. *The optimal mechanism is stationary and involves trading immediately upon entry whenever $v \geq \underline{v}$ and $c \leq \bar{c}$:*

$$q_{i,s}^B(v) = \begin{cases} 1, & \text{if } v \geq \underline{v} \text{ and } s = \tau(i), \\ 0, & \text{otherwise,} \end{cases} \qquad q_{j,s}^S(c) = \begin{cases} 1, & \text{if } c \leq \bar{c} \text{ and } s = \tau(j), \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

where the marginal types $\underline{v}$ and $\bar{c}$ are determined by

$$J_B(\underline{v}) = J_S(\bar{c}), \qquad 1 - F_B(\underline{v}) = F_S(\bar{c}).$$

The proof is rather long and is given in Appendix A. That the optimal mechanism is stationary and involves trading immediately upon entry in a two-sided private information setup is to be expected, since a similar result was obtained in one-sided setups (see the literature on durable goods in the context of a large market, for example, Stokey (1979), Segal (2003), Board (2008)). The optimal price p_S set by a profit maximizing seller with cost c is given by $J_B(p_S) = c$ according to Myerson (1981).[13] Similarly, a profit maximizing buyer would set the price given by $J_S(p_B) = v$. Putting the optimal buying and selling strategy and the goods market balance condition together determines the marginal types.[14] As in Bulow and

[13] While the argument in Myerson (1981) is more subtle, a simplified version is that $J_B(p_S) = c$ is the first order condition when maximizing $(p_S - c)(1 - F_B(p_S))$ with respect to p_S.

[14] See also Baliga and Vohra (2003) and Loertscher and Niedermayer (2012) for the optimal mechanism with a discrete number of traders in a one-period model.

Roberts (1989), J_B can be seen as marginal revenue and J_S as marginal cost. The intermediary increases the quantity traded, $q = 1 - F_B(\underline{v}) = F_S(\overline{c})$, as long as marginal revenue is higher than marginal cost, $J_B(F_B^{-1}(1-q)) > J_S(F_S^{-1}(q))$, and stops as soon as it is lower.

The implied profit maximizing spread $\theta^* = \underline{v} - \bar{c}$ is given by

$$\theta^* = \frac{1 - F_B(\underline{v})}{f_B(\underline{v})} + \frac{F_S(\bar{c})}{f_S(\bar{c})} = \frac{\underline{v}}{\eta_B(\underline{v})} + \frac{\bar{c}}{\eta_S(\bar{c})}, \tag{4}$$

where the $\eta_B(v) = v f_B(v)/(1-F_B(v))$ and $\eta_S(c) = -c f_S(c)/F_S(c)$ are the elasticities of demand and supply, respectively, and $v/\eta_B(v)$ and $c/\eta_S(c)$ are the inverse semi-elasticities.

There is a straightforward centralized implementation of the allocation rule described in Proposition 1. The intermediary sets an ask price $\underline{v}$ and a bid price $\overline{c}$ in each period. Buyers with valuation $v \geq \underline{v}$ and sellers with cost $c \leq \overline{c}$ trade, all others do not. The intermediary earns the bid-ask spread $\underline{v} - \overline{c}$.

4 Decentralized Implementation: Bargaining with Participation Fees

The implementation using the fixed spread mechanism mentioned in the previous section is fully centralized. As explained in the Introduction, in many markets an intermediary might find it too costly to implement a centralized mechanism. In this section, we turn to the main question of our paper: a decentralized implementation of the optimal allocation.

We consider a *participation fee mechanism*: charge buyers and sellers per period participation fees K_B and K_S in a dynamic random matching environment with bargaining. We will show that such a mechanism can generate the same profits as the optimal mechanism. For the moment, we focus on per period fees. In Section 5.1 we show how results are changed when looking at per match or per transaction fees.

Each period $\tau = 0, 1, 2, \ldots$ of the dynamic matching and bargaining game is composed of the following stages:

1. Potential entrants arrive to the market and make entry decisions. Those who have chosen to enter, are added to the pool of active traders carried over from the previous period.

2. Each active trader pays the broker his or her participation fee, K_B for buyers and K_S for sellers.

3. Traders currently in the unmatched pool are randomly matched in pairs of one buyer and one seller. We assume that the short side of the market is fully matched, so that the mass of matches is given by $\min\{B, S\}$, where B and S are the masses of active buyers and active sellers currently in the market.

4. Matched buyers and sellers bargain according to the random-proposer, take-it-or-leave-it protocol. The buyer gets to make a take-it-or-leave-it price offer with probability α_B. With probability $\alpha_S = 1 - \alpha_B$ the seller gets to make the offer. If a type v buyer and a type c seller trade at a price p, then they leave the market with payoff $v - p$ and $p - c$, respectively.

5. Unsuccessful traders carry over to the next period.

Note that the proposal probabilities should be seen as a stylized way of modeling bargaining strength. The two extreme cases of $\alpha_S = 1$ and $\alpha_B = 1$ can be taken literally. In the former case, the price is always set by the seller (for example, shops in malls), in the latter case, the price is always set by the buyer (this is a reasonable approximation for markets for low skilled labor). Further, note that instead of assuming that one side is randomly chosen to make a take-it-or-leave-it offer, one could also assume that one side is randomly chosen to design the mechanism for bilateral trade. By Myerson (1981), Yilankaya (1999), and Mylovanov and Tröger (2012), the party designing the mechanism would choose a take-it-or-leave-it offer as the mechanism.[15]

We assume the market is anonymous from the perspective of traders, so the traders do not know their partners' market history, for example, how long they have been in the market, what they proposed previously, and what offers they rejected previously. Anonymity from the perspective of the intermediary is immaterial, since the optimal mechanism does not condition on identity or date of entry. We restrict attention to steady-state market equilibria, defined in parallel to Satterthwaite and Shneyerov (2007) and Shneyerov and Wong (2010a,b). Since the optimal mechanism has the same allocation rule in every period, the intermediary could not get higher profits if he were to choose fees that induce a non-steady state equilibrium.

[15]If the seller is to design the mechanism and her type were public information, it is well known from Myerson (1981) that her optimal mechanism would be a take-it-or-leave-it offer. In a setup where the seller's type is private information Yilankaya (1999), and more generally Mylovanov and Tröger (2012), show that if the mechanism is endogenously chosen by the seller rather than exogenously given, the optimal mechanism is the same as if the seller's type were public information.

Let $\chi_B : [0,1] \to \{0,1\}$ and $\chi_S : [0,1] \to \{0,1\}$ be the buyers' and sellers' entry strategies, with the value of 1 assigned to entry. Let $A_B \subset [0,1]$ and $A_S \subset [0,1]$ be the sets of active buyers' and sellers' types, that is,

$$A_B := \{v \in [0,1] : \chi_B(v) = 1\}, \qquad A_S := \{c \in [0,1] : \chi_S(c) = 1\}.$$

Let $p_B(v)$ and $p_S(c)$ be the proposing strategies used by buyers and sellers respectively. Let $\tilde{v}(v)$ and $\tilde{c}(c)$ denote the responding strategies of buyers and sellers. More precisely, when proposing, for example, type v buyers will propose the trading price $p_B(v)$, while when responding, they will accept a proposed price p if and only if $\tilde{v}(v) \geq p$. Let $\Phi_B(v), \Phi_S(c)$ be the (endogenous) steady-state cumulative distributions of types of buyers and sellers who are active.

Before we can formally define a steady state market equilibrium, consider the sequential optimality of the responding strategies first. Let $W_B(v)$ be the (steady-state) equilibrium continuation payoff of a type v buyer at the period's beginning, and let $W_S(c)$ be the equilibrium continuation payoff of a type c seller. Pick a type v buyer.[16] *Sequential optimality* requires that a buyer accept any offer lower than his *dynamic value* $v - \delta W_B(v)$, and a seller accept any offer higher than her dynamic cost $c + \delta W_S(c)$.[17] These *dynamic types* determine traders' responding strategies.

Consider the distributions of traders' dynamic types, denoted as

$$\tilde{\Phi}_B(x) := \int_{v-\delta W_B(v)\leq x} d\Phi_B(v), \qquad \tilde{\Phi}_S(x) := \int_{c+\delta W_S(c)\leq x} d\Phi_S(c). \tag{5}$$

It is clear that sequential optimality in the proposing states is satisfied if and only if

$$p_B(v) \in \arg\max_{\lambda} \tilde{\Phi}_S(\lambda)[v - \delta W_B(v) - \lambda], \qquad p_S(c) \in \arg\max_{\lambda} [1 - \tilde{\Phi}_B(\lambda)][\lambda - c - \delta W_S(c)]. \tag{6}$$

It follows that the equilibrium proposing strategies are determined as best-responses in the static monopoly problems where the distributions of responders' types are replaced by the distributions of the responders' dynamic types and the proposers' types are replaced by the proposers' dynamic types.

To introduce the Bellman equations for the continuation values, consider a buyer who is matched with a seller. Suppose that all traders always use their prescribed equilibrium

[16] This type v buyer could be either active or not. If he is not active, we are considering an off-equilibrium path.

[17] Dynamic (opportunity) values and costs are introduced in Satterthwaite and Shneyerov (2007).

strategies and that the stationary distributions of active seller and buyer types are at their equilibrium values Φ_S and Φ_B. Then a type v buyer's expected bargaining surplus from the meeting is equal to

$$\Pi_B(v) := \alpha_B \int_{c+\delta W_S(c) \leq p_B(v)} [v - p_B(v)] d\Phi_S(c) + \alpha_S \int_{p_S(c) \leq v - \delta W_B(v)} [v - p_S(c)] d\Phi_S(c).$$

Further denote

$$q_B(v) := \alpha_B \int_{c+\delta W_S(c) \leq p_B(v)} d\Phi_S(c) + \alpha_S \int_{p_S(c) \leq v - \delta W_B(v)} d\Phi_S(c),$$

the buyer's probability of a successful trade in a given meeting.

Since the entry decision is made in the inactive state and a trader gets 0 if he exits, the beginning-of-period continuation payoff, $W_B(v)$, must satisfy the following recursive equation:

$$W_B(v) = \max \left\{ \frac{\min\{B, S\}}{B} [\Pi_B(v) + (1 - q_B(v))\delta W_B(v)] - K_B, 0 \right\} \tag{7}$$

where the first maximand represents the payoff for entry, the second represents the payoff for exiting. Therefore, the buyers' sequentially optimal entry strategy is

$$\chi_B(v) = I \left\{ \frac{\min\{B, S\}}{B} \Pi_B(v) \geq K_B \right\} \tag{8}$$

where $I(\cdot)$ is the indicator function.[18] Completely parallel logic applies to the sellers' side; we have the Bellman equation

$$W_S(c) = \max \left\{ \frac{\min\{B, S\}}{S} [\Pi_S(c) + (1 - q_S(c))\delta W_S(c)] - K_S, 0 \right\}, \tag{9}$$

with a seller's expected bargaining surplus from the meeting

$$\Pi_S(c) := \alpha_S \int_{v-\delta W_B(v) \geq p_S(c)} [p_S(c) - c] d\Phi_B(v) + \alpha_B \int_{c+\delta W_S(c) \leq p_B(v)} [p_B(v) - c] d\Phi_B(v),$$

and the sellers' sequentially optimal entry strategy is

$$\chi_S(c) = I \left\{ \frac{\min\{B, S\}}{S} \Pi_S(c) \geq K_S \right\}. \tag{10}$$

To complete the description of a (nontrivial[19]) steady-state equilibrium, we now consider the steady state equations for the distributions of active buyer and seller types Φ_B and Φ_S and

[18] Note that (8) implicitly assumes that traders enter if they are indifferent between entering or not. This is only for expositional simplicity because it turns out that the set of such indifferent traders is of measure 0.

[19] In a trivial equilibrium, no buyer and no seller enters, which implies that it is optimal for both sellers and buyers not to enter.

active trader masses B and S. In a steady-state market equilibrium, traders who enter would not exit until they have traded successfully. Therefore,

$$\int_v^1 \chi_B(x)dF(x) = \min\{B,S\}\int_v^1 q_B(x)d\Phi_B(x), \quad \forall v \in [0,1], \tag{11}$$

$$\int_0^c \chi_S(x)dG(x) = \min\{B,S\}\int_0^c q_S(x)d\Phi_S(x), \quad \forall c \in [0,1], \tag{12}$$

which simply state that the inflow mass of every type of a trader must be equal to the corresponding outflow mass.

These preparations allow us to formally define a nontrivial steady-state equilibrium as follows.

Definition 1. *A collection $E := \{\chi_B, \chi_S, p_B, p_S, B, S, \Phi_B, \Phi_S\}$ is a nontrivial steady-state equilibrium if there exists a pair of equilibrium payoff functions $\{W_B, W_S\}$ such that the proposing strategies p_B, p_S and entry strategies χ_B, χ_S satisfy the sequential optimality conditions* (6)*,* (8)*, and* (10)*, and the distributions of active buyer and seller types Φ_B and Φ_S and active trader masses B and S solve the steady-state equations* (11) *and* (12)*, and the payoff functions W_B and W_S solve the Bellman equations* (7) *and* (9)*.*

It can be shown that in any such equilibrium, buyers above some threshold $\underline{v}$ and sellers below some threshold $\overline{c}$ are active. Non-active traders stay out of the market. This is stated formally in the following Lemma.

Lemma 1. *In any equilibrium, the set of active buyers is $[\underline{v}, 1]$ and the set of active sellers is $[0, \overline{c}]$ for some $\underline{v}, \overline{c} \in [0,1]$.*

Proof. The proof follows the same logic as Lemma 1 in Shneyerov and Wong (2010b) and therefore omitted. □

Lemma 1 allows us to focus our analysis on the marginal types $\underline{v}$ and $\overline{c}$. The marginal buyer $\underline{v}$ gets a zero net expected utility from participating, hence his option value of future trade is also zero. Further, a seller would never set a price below $\underline{v}$, hence the buyer's utility if the seller makes the offer is zero as well. Therefore, when looking for the marginal buyer we only need to consider the buyer's utility in case he makes the offer, which gives us

$$\frac{\min\{B,S\}}{B}\alpha_B\Phi_S(\underline{p}_B)(\underline{v} - \underline{p}_B) = K_B, \tag{13}$$

where $\underline{p}_B$ is the optimal price set by the marginal buyer, and $\Phi_S(\cdot)$ is the steady-state distribution of seller types in the market. The left-hand side of the equation is the gross utility of participation of the marginal buyer: the product of the probability of being matched $\min\{B,S\}/B$, the probability of making the offer α_B, of the seller accepting the offer $\Phi_S(\underline{p}_B)$, and the buyer's utility when buying at the price the marginal buyer sets $\underline{v}-\underline{p}_B$. The right-hand side is the cost of participation K_B. Similarly, for the marginal seller $\bar{c}$

$$\frac{\min\{B,S\}}{S}\alpha_S(1-\Phi_B(\overline{p}_S))(\bar{c}-\overline{p}_S)=K_S, \tag{14}$$

where $\Phi_B(\cdot)$ is the steady-state distribution of buyer types.

While deriving the market equilibrium is complicated in general in such setups, since there may be multiple equilibria, a full-trade equilibrium may not exist for all values of K_B and K_S (see, for example, Shneyerov and Wong (2010a)), and almost everything in (13) and (14) is endogenous, it turns out that the analysis is strongly simplified by focusing on the profit-maximizing equilibrium.

We know from the analysis in the previous section that if an intermediary can implement the allocation rule of the centralized mechanism, then he cannot do better. In the following we will show that choosing K_B and K_S indeed enables the intermediary to do this. We will hence focus our attention on the optimal allocation rule, which has the properties that there is full trade (anyone who gets matched, trades with probability 1, here: $\underline{p}_B=\bar{c}$ and $\overline{p}_S=\underline{v}$) and that the market is balanced $1-F_B(\underline{v})=F_S(\bar{c})$ (therefore $\min\{B,S\}=B=S$). We will first characterize the fee structure that is implied by the marginal types in a balanced full trade equilibrium. Then we will show, in Proposition 2 below, that this fee structure indeed induces a balanced full-trade equilibrium.

Denote the optimal fees as K_B^* and K_S^*. For full-trade balanced market equilibria, the conditions for the marginal types (13) and (14) reduce to

$$\alpha_B\theta=K_B^*, \tag{15}$$

$$\alpha_S\theta=K_S^*, \tag{16}$$

where $\theta=\underline{v}-\bar{c}$ is the spread. This is due to the balanced market $\min\{B,S\}/B=\min\{B,S\}/S=1$ and full trade $\Phi_S(\underline{p}_B)=1-\Phi_B(\overline{p}_S)=1$. Dividing the first with the second equation gives us

$$\frac{\alpha_B}{\alpha_S}=\frac{K_B}{K_S}. \tag{17}$$

Adding the two equations gives us

$$\theta^* = K_B^* + K_S^*,$$

where θ^* is the profit maximizing spread chosen by the intermediary. Following the remark after (4) this can also be written in terms of the elasticities

$$K_B^* + K_S^* = \frac{\underline{v}}{\eta_B(\underline{v})} + \frac{\bar{c}}{\eta_S(\bar{c})}, \tag{18}$$

where $\underline{v}$ and $\bar{c}$ are given by the optimal allocation rule. These results mean that the sum of the fees is equal to the sum of the semi-elasticities of demand and supply for marginal traders. It further means that the ratio of the fees (the price structure) is independent of the elasticities and is equal to the ratio of the bargaining weights. Therefore, the side of the market with the stronger bargaining power will be charged a higher fee. Take, for example, the special case where the sellers set the price (that is, $\alpha_S = 1$). In this case participation is free for buyers, $K_B^* = 0$, and sellers bear the full burden of the fee, $K_S^* = \theta^*$.

Existence of a full-trade equilibrium. Before showing existence of a balanced full-trade equilibrium in a dynamic setup, the following intuition for existence can be given in the simpler static setup with $\delta = 0$. A seller faces the following trade-off when considering raising the price: a higher price increases profits in case of trade, but it also decreases the probability of trade. An intermediary faces the same trade-off, but can additionally lower his costs as he raises the price for buyers. This is because less entry on the buyer side means that he can decrease the number of sellers entering while keeping the trade volume constant, which lowers his cost. Hence, if the intermediary is not willing to deviate from full trade in the centralized optimal mechanism, neither are the sellers in the decentralized setup. The same reasoning applies to the buyers.

Returning to the dynamic case (arbitrary δ), recall that the profit maximizing spread of the intermediary is given by

$$J_B(\underline{v}) = J_S(\bar{c}). \tag{19}$$

It can be shown that if the spread between $\underline{v}$ and $\bar{c}$ is large enough, then a full-trade equilibrium exists. Intuitively, the reason is that the marginal seller $\bar{c}$ does not have an interest to set a price above $\underline{v}$ if the difference $\underline{v} - \bar{c}$ is large enough. The same applies to the marginal buyer $\underline{v}$. We show this formally in the following Lemma.

Lemma 2. *A full-trade equilibrium exists if the marginal types satisfy the following conditions:*

$$(1-\delta)J_B(\underline{v}) + \delta\underline{v} \geq \bar{c}, \tag{20}$$

$$(1-\delta)J_S(\bar{c}) + \delta\bar{c} \leq \underline{v}. \tag{21}$$

To gain an intuition for this result consider the static setup, $\delta = 0$. The expected profit of a seller of type c in any match where she deviates from equilibrium and proposes an ask price p_S is

$$\begin{aligned}\pi_S(c, p_S) :&= \alpha_S(p_S - c)(1 - \Phi_B(p_S)) + \alpha_B(\bar{c} - c) \\ &= \alpha_S(p_S - c)\frac{1 - F_B(p_S)}{1 - F_B(\underline{v})} + \alpha_B(\bar{c} - c), \\ &= \alpha_S\frac{\int_{p_S}^1 (J_B(v) - c)dF_B(v)}{1 - F_B(\underline{v})} + \alpha_B(\bar{c} - c),\end{aligned} \tag{22}$$

where the last equality can be checked by verifying that both $(p_S - c)(1 - F_B(p_S))$ and $\int_{p_S}^1 (J_B(v) - c)dF_B(v)$ are 0 for $p_S = 1$ and that

$$\frac{\partial[(p_S - c)(1 - F_B(p_S))]}{\partial p_S} = -(J_B(p_S) - c)f_B(p_S).$$

The slope of the expected profit is

$$\frac{\partial \pi_S(c, p_S)}{\partial p_S} = -\frac{\alpha_S(J_B(p_S) - c)f_B(p_S)}{1 - F_B(\underline{v})}. \tag{23}$$

At $p_S = \underline{v}$, this slope is non-positive for all $c \leq \bar{c}$ if $J_B(\underline{v}) \geq \bar{c}$, which implies that the seller will not prefer such a deviation. Intuitively, (23) is the seller's marginal surplus of trading with a buyer with valuation $v = p_S$ if she gets to make the offer. Since this marginal surplus is always positive, increasing the price p_S (that is, excluding buyers), lowers profits. $J_B(\underline{v}) \geq \overline{c}$ is our condition (20) in the statement of the lemma when $\delta = 0$.

An intuition can be found for the other extreme as well, that is, $\delta \to 1$ (infinitely patient traders) rather than $\delta \to 0$. Observe that demand becomes infinitely elastic in the limit at $p = \underline{v}$, since it is (almost) costless for a buyer to delay trade and wait for a better offer. Therefore, the marginal seller will not offer a price above $\underline{v}$. Neither will he offer a price below $\overline{c}$. Therefore, a full-trade equilibrium exists if $\underline{v} \geq \overline{c}$ as stated in (20) for $\delta \to 1$. The intuition for (21) is analogous.

Proof of Lemma 2. We only need to show that sellers do not have an incentive to deviate to an ask price higher than $\underline{v}$, and buyers do not have an incentive to deviate to a bid price lower than $\bar{c}$. We will only prove the result for sellers; for buyers, the argument is parallel.

The probability that an offer p_S is accepted depends on the market distribution of buyers' *dynamic values.* Consider a buyer of type v. His expected profit in the next market opening, discounted back to the current one, is equal to

$$\begin{aligned}
W_B(v) &= \delta\left(\alpha_B(v-\bar{c}) + \alpha_S(v-\underline{v}) - K_B^*\right), \\
&= \delta(\alpha_B+\alpha_S)(v-\underline{v}) + \delta\alpha_B(\underline{v}-\bar{c}) - \delta K_B^* \\
&= \delta(v-\underline{v}), \qquad (24)
\end{aligned}$$

where the equality in the last line follows from the fact that, in a full-trade equilibrium, $\alpha_B(\underline{v}-\bar{c}) = \alpha_B\theta = K_B$. In the current period, he will accept any price p_S if $v - p_S \geq W_B(v)$. Equivalently, he will accept any p_S at or above his dynamic value

$$\tilde{v}(v) := v - W_B(v) = (1-\delta)v + \delta\underline{v}. \qquad (25)$$

Going in the reverse direction, the buyer's type v as a function of his dynamic value $\tilde{v}$ is

$$v = \frac{\tilde{v}-\delta\underline{v}}{1-\delta},$$

and therefore the market distribution of dynamic values is

$$\tilde{\Phi}_B(\tilde{v}) = \Phi_B\left(\frac{\tilde{v}-\delta\underline{v}}{1-\delta}\right).$$

The expected seller's profit in any match is of the same form as in (22), with $\Phi_B(p_S)$ replaced with $\tilde{\Phi}_B(p_S)$. The corresponding virtual valuations are

$$\begin{aligned}
\tilde{J}_B(\tilde{v}) : &= \tilde{v} - \frac{1-\tilde{\Phi}_B(\tilde{v})}{\tilde{\phi}_B(\tilde{v})} \\
&= (1-\delta)v + \delta\underline{v} - (1-\delta)\frac{1-\Phi_B(v)}{\phi_B(v)} \\
&= (1-\delta)v + \delta\underline{v} - (1-\delta)\frac{1-F_B(v)}{f_B(v)} \qquad (26) \\
&= (1-\delta)J_B(v) + \delta\underline{v}. \qquad (27)
\end{aligned}$$

In parallel to the reasoning the static setup which leads from (22) to (23), $\partial\pi_S(c,\underline{v})/\partial p_S$ has the same sign as $-(\tilde{J}_B(\underline{v})-c)$, which equals $-((1-\delta)J_B(\underline{v})+\delta\underline{v}-c)$, so (27) implies (20) in the lemma. □

We can use Lemma 2 to show existence of a profit maximizing equilibrium. Using the profit maximizing spread in (19) and $J_S(c) \geq c$ we can find a lower bound for the left-hand side of

(20) in Lemma 2:

$$(1-\delta)J_B(\underline{v}) + \delta\underline{v} = (1-\delta)J_S(\bar{c}) + \delta\underline{v} \geq (1-\delta)\bar{c} + \delta\underline{v}$$

which is greater or equal $\bar{c}$, since $\underline{v} \geq \bar{c}$ for a profit maximizing intermediary. Hence, condition (20) is always fulfilled for the profit maximizing spread. By an analogous reasoning, condition (21) is also always satisfied. Since the ratio of the fees K_B/K_S is chosen such that a balanced market is achieved if there is full trade, we have shown existence of a full-trade balanced market equilibrium that maximizes the intermediary's profit.

Proposition 2. *The intermediary's profit maximizing per period fees K_B^* for the buyer and K_S^* for the seller in the dynamic random matching setup are given by*

$$K_B^* + K_S^* = \frac{1-F_B(\underline{v})}{f_B(\underline{v})} + \frac{F_S(\overline{c})}{f_S(\overline{c})},$$
$$\frac{K_B^*}{K_S^*} = \frac{\alpha_B}{\alpha_S},$$

where the marginal types $\underline{v}$ and $\overline{c}$ are given by Proposition 1. For these fees, a balanced full-trade market equilibrium exists in the dynamic random matching setup, that is, an equilibrium such that $\min\{B,S\}/B = \min\{B,S\}/S = 1$ and $\Phi_S(\underline{p}_B) = 1 - \Phi_B(\overline{p}_S) = 1$, where $\underline{p}_B$ is the lowest price set by a buyer and $\overline{p}_S$ is the highest price set by a seller.

Proposition 2 tells us that the insights from a static setup described at the beginning of this section carry over to a dynamic setup. The reason why we learn something new from the dynamic setup is that traders face different dynamic incentives than the intermediary. In particular, the intermediary internalizes the fact that a buyer who does not transact today will be present tomorrow, whereas a seller does not internalize this. However, this provides further incentives to a seller to set a lower price than the intermediary. Therefore, dynamics reinforces the forces at work in the static setup.

5 Discussion

5.1 Other Types of Fees

We now consider two other types of fees: per match fees and transaction fees.

Per Match Fees Per match fees are only paid if a trader is matched. The only difference between participation fees and per match fees is that per match fees do not have to be paid if a trader participates in the market, but is not matched to anyone in a certain period. However, since the optimal equilibrium chosen by the intermediary is such that the market is balanced, a match occurs with probability 1. Therefore, payments are the same for per match fees and participation fees on the equilibrium path. Payments off the equilibrium path do not change for individual traders either, since the probability of being matched is independent of a trader's action. In particular, both per match fees and participation fees are sunk at the time of bargaining and therefore affect the bargaining behavior the same way. Therefore, the same reasoning can be applied to per match fees as for participation fees and by analogy to Proposition 2 we get

$$\hat{K}_B + \hat{K}_S = \frac{1 - F_B(\underline{v})}{f_B(\underline{v})} + \frac{F_S(\overline{c})}{f_S(\overline{c})}, \qquad \frac{\hat{K}_B}{\hat{K}_S} = \frac{\alpha_B}{\alpha_S},$$

where $\hat{K}_B$ and $\hat{K}_S$ are the fees to be paid upon a match by the buyer and seller, respectively.

Transaction Fees Transaction fees are only paid if a transaction occurs. The crucial difference between transaction fees on one hand and per match fees and participation fees on the other hand is that transaction fees are not sunk at the time of bargaining. Therefore, they affect the deviation incentives of traders. In the following, we will show that in our setup the optimal mechanism cannot be implemented by charging only transaction fees, as shown in part (i) of Proposition 3. While transaction fees alone are not sufficient to implement the optimal mechanism, a combination of participation fees and transaction fees still is, provided that transaction fees are not too large as shown in part (ii) of Proposition 3.

Proposition 3. *(i) The outcome of the optimal mechanism cannot be implemented in a dynamic matching and bargaining game with transaction fees only.*

(ii) However, it can be implemented by dynamic matching and bargaining and a combination of participation fees K_B, K_S and transaction fees κ_B, κ_S with

$$\underline{v} - \overline{c} = K_B + K_S + \kappa_B + \kappa_S, \qquad \frac{\alpha_B}{\alpha_S} = \frac{K_B}{K_S}, \tag{28}$$

where the sum of transaction fees has to satisfy

$$\kappa_B + \kappa_S \leq \min\{\tilde{J}_B(\underline{v}) - \overline{c}, \underline{v} - \tilde{J}_S(\overline{c})\}. \tag{29}$$

In the resulting balanced full-trade equilibrium, sellers set the price $\overline{p}_S = \underline{v} - \kappa_B$ *and buyers set the price* $\underline{p}_B = \overline{c} + \kappa_S$.

The proof is provided in Appendix A. The basic intuition for part (i) of the Proposition is the following. The marginal seller has a zero utility if she sets a price that makes the marginal buyer indifferent. If she increases her price by an epsilon, she gets a strictly positive utility if her offer is accepted (which happens with positive probability). If her offer is rejected, she does not have to pay any fees (this is different from participation fees and per match fees), so she gets a non-negative utility by deferring (potential) trade to the future. Therefore, increasing the price is a profitable deviation for the seller. This implies that the balanced full-trade equilibrium of the optimal mechanism is not implementable.

An intuition for part (ii) of the Proposition is that if transaction fees are sufficiently low, marginal traders make a sufficiently large profit from setting a price that is accepted with probability 1, taking into account that their participation fees are already sunk. This makes it unprofitable for marginal traders to deviate from a full-trade equilibrium.

It is easy to check that for $\kappa_B + \kappa_S = 0$, we are back to the basic participation fee setup. Note that for $\delta \to 1$, $\tilde{J}_B(\underline{v}) - \overline{c} \to \underline{v} - \overline{c}$ and $\underline{v} - \tilde{J}_S(\overline{c}) \to \underline{v} - \overline{c}$ in (29), so that by (28) the intermediary can earn most of the spread $\underline{v} - \overline{c}$ in form of transaction fees and charge participation fees close to zero (but strictly positive[20] and satisfying $K_B/K_S = \alpha_B/\alpha_S$).

In our setup, transaction fees do not matter much: the intermediary can avoid charging transaction fees and simply charge participation fees. Alternatively, the intermediary can reduce participation fees and charge the reduced amount as transaction fees (as long as these are not too large). In reality, we observe some examples for which an intermediary charges both participation fees and transaction fees. An example are credit card companies that charge both annual membership (or participation) fees to merchants and consumers and also charge transaction fees. The reason why an intermediary would want to "convert" some of the participation fees in transaction fees have to lie outside of our model.[21]

[20] More precisely, the fee is strictly positive on at least one side of the market: for the special case $\alpha_i = 0$ on one side, the participation fee is $K_i = 0$ on that side.

[21] One reason is described in the essentially static setup in Loertscher and Niedermayer (2012): in thin markets (that is, there is only one buyer and one seller rather than a continuum on both sides as here) an intermediary prefers transaction fees and a non-full-trade equilibrium to participation fees and a full-trade equilibrium.

5.2 Exogenous Participation Costs

The model can be easily extended to include both participation fees K_B and K_S set by the intermediary and exogenously given per period search costs x_B and x_S per period. The conditions for the marginal types are

$$\alpha_B\theta^* = K_B + x_B \qquad \alpha_S\theta^* = K_S + x_S,$$

which is essentially the same as before, except that total participation costs $K_B + x_B$ and $K_S + x_S$ are decomposed to fees and exogenous search costs. The optimal spread is $\underline{v} - \overline{c} = \theta^* = \underline{v}/\eta_B(\underline{v}) + \overline{c}/\eta_S(\overline{c}) + x_B + x_S$ and the market balancing condition becomes $\alpha_B/\alpha_S = (K_B + x_B)/(K_S + x_S)$. As an example, in a market where sellers always get to make take-it-or-leave-it offers to buyers ($\alpha_S = 1$) and buyers have positive exogenous search costs ($x_B > 0$), the platform should charge high fees to sellers ($K_S = \theta^* - x_S + x_B$) and subsidize buyers ($K_B = -x_B$).

5.3 Comparison with the Two-Sided Markets Literature

It is interesting to compare our results about the fees charged by the platform with those in the two-sided markets literature. Most of the two-sided markets literature is concerned with a reduced form approach, in which the utility of the buyer directly depends on the number of sellers and the utility of the seller directly depends on the number of buyers. Some of the articles in the two-sided markets literature also provide a random-matching microfoundation for the reduced form utility function.

In the following, we will compare the predictions of our model with the prediction of the articles with a random-matching microfoundation and discuss the differences in assumptions. We will make the comparison with Rochet and Tirole (2006), who assume in the (static) random-matching microfoundation part of their paper (p. 661-663) that buyers and sellers do not know their types when choosing whether to enter the market and that the matching technology is increasing returns to scale (that is, the mass of matches is BS).

In Rochet and Tirole (2006, Proposition 2) the sum of the optimal overall fees[22] $\tilde{K}_B$ and $\tilde{K}_S$ is given by

$$\tilde{K}_B + \tilde{K}_S = \frac{\tilde{K}_B}{\eta_B} = \frac{\tilde{K}_S}{\eta_S}. \tag{30}$$

[22] In Rochet and Tirole (2006) the overall fee is composed of a participation and a transaction fee.

The difference is mainly due to an increasing returns to scale rather than a constant returns to scale matching technology in their model. A far more interesting comparison is that of the ratio of fees. This is given in Rochet and Tirole (2006) as

$$\frac{\tilde{K}_B}{\tilde{K}_S} = \frac{\eta_B}{\eta_S}, \tag{31}$$

that is, by the ratio of the elasticities. One has to be careful with the interpretation of (31), since η_B and η_S are endogenous.[23] In our setup, the fee structure is independent of the elasticities of demand, but only depends on the relative bargaining weights, $K_B/K_S = \alpha_B/\alpha_S$.

The following intuition can be given for this difference in predictions. In Rochet and Tirole (2006) buyers and sellers find out their valuations for the good *after* deciding whether to join the platform. Therefore, the platform can separately solve two problems: how the overall fees affect participation by traders and how transaction fees affect the efficiency of trade. The effect of the overall fee on the participation of traders can be seen as two separate monopoly pricing problems: choosing $\tilde{K}_B$ optimally given $\tilde{K}_S$ and choosing $\tilde{K}_S$ optimally given $\tilde{K}_B$. For choosing $\tilde{K}_i$, (30) can also be written as

$$\tilde{K}_i - (-\tilde{K}_{-i}) = \frac{K_i}{\eta_i}, \qquad i = B, S, \quad -i = S, B, \tag{32}$$

which is the Lerner formula for monopoly pricing, $p - c = p/\eta$ (that is, markup equal to semi-elasticity), with price $p = K_i$ and costs $c = -K_{-i}$.

At the second stage, after traders have decided about entry and subsequently found out their types, the platform can set transaction fees with efficiency considerations in mind.

In our setup, however, agents find out their types *before* deciding whether to enter the platform. Therefore, the problem of influencing entry decision and the efficiency of trade cannot be separated. The idea is best illustrated with the extreme case $\alpha_S = 1$, that is, the seller sets the price. If the buyer were to pay a positive fee $K_B > 0$, the marginal buyer $\underline{v}$ would be held up after making the decision to enter the platform and paying K_B: a seller would never set a price below $\underline{v}$. Therefore, the marginal buyer would have a zero gross utility and a negative utility $-K_B$ when taking the fee into account. This causes an unraveling of the market, since no matter who the lowest valuation buyer is, he will anticipate ex ante that he will make a loss and refuses to enter. Therefore, the optimal buyer fee is $K_B = 0$, or $K_B/K_S = \alpha_B/\alpha_S = 0$.

[23] The interpretation of this is more subtle than it might appear at first sight: the equality holds in equilibrium at the optimal fees and optimal elasticities. It should not be interpreted as charging a higher fee to the *more* elastic side of the market (see Krueger (2009) for a detailed discussion of this issue).

In general (for arbitrary values of α_S), the sum of fees is equal to the sum of semi-elasticities,

$$K_B + K_S = \frac{\underline{v}}{\eta_B} + \frac{\overline{c}}{\eta_S},$$

in our setup. The difference from (32) reflects the fact that the problem of influencing the entry decision and the efficiency of trade cannot be separated in our setup.

5.4 Welfare Effects of Intermediaries

Our model can be used to shed light on the debate whether intermediaries reduce welfare. While a detailed analysis is outside of the scope of this paper, we add a brief discussion of these issues in light of our theoretical framework. In particular, we ask the following question: would welfare increase or decrease if the intermediary were removed from the market and participation were made free?

In this section, we compare a market without fees with the same market run by a for-profit intermediary. We will use the Walrasian outcome as a benchmark, in which buyers with types above and sellers below the Walrasian price p^* trade and the Walrasian price is given by $1 - F_B(p^*) = F_S(p^*)$.

To simplify the exposition, we mostly focus on two extreme cases. In one extreme case the discount factor is $\delta \to 0$. This is essentially a one-shot game, since only the first period profits of the intermediary and the first period utilities of traders matter. The case $\delta \to 0$ can be viewed as search frictions being large. The other extreme is $\delta \to 1$, that is, market participants are infinitely patient. This can be viewed as search frictions disappearing in the limit.

For an analytically tractable example, we assume that traders' types follow Generalized Pareto distributions, $F_B(v) = 1 - (1-v)^\beta$ and $F_S(c) = c^\beta$ with $\beta > 0$. Generalized Pareto distributions result in linear virtual types, $J_B(v) = (1 + \frac{1}{\beta})v - \frac{1}{\beta}$ and $J_S(c) = (1 + \frac{1}{\beta})c$.[24]

Walrasian Allocation The Walrasian price is $p^* = \frac{1}{2}$ because of symmetry. Buyers' welfare is $\sum_{s=0}^{\infty} \delta^s \int_{p^*}^{1} (v - p^*) dF_B(v)$. To allow for a comparability across different values of δ, we normalize by multiplying by $(1-\delta)$, which yields the normalized welfare

$$W_B^* = (1-\delta) \sum_{s=0}^{\infty} \delta^s \int_{p^*}^{1} (v - p^*) dF_B(v) = \frac{1}{(\beta+1)2^{\beta+1}},$$

[24] A higher β can be seen as a higher elasticity of demand and supply, since the elasticities are $\eta_B(v) = \beta v/(1-v)$ and $\eta_S(c) = \beta$. Further, as $\beta \to 0$, informational asymmetries vanish, since in the limit buyers have valuation 1 with probability 1 and sellers have valuation 0 with probability 1.

for buyers. Similarly,

$$W_S^* = \int_0^{p^*} (p^* - c) dF_S(c) = \frac{1}{(\beta+1)2^{\beta+1}}.$$

For-Profit Platform Let us next consider welfare when the market is run by a for-profit platform charging participation fees as described in our model. By our revenue equivalence results, the payoffs of buyers, sellers, and the intermediary are the same with participation fees and with bid and ask prices $\underline{v}$ and $\overline{c}$. We can hence simplify the analysis by looking at payoffs of a market maker setting bid-ask prices. Here, buyers above $\underline{v}$ and sellers below $\overline{c}$ enter and trade occurs at price $\underline{v}$ and $\overline{c}$. The marginal types satisfy $1 - F_B(\underline{v}) = F_S(\overline{c})$ and $J_B(\underline{v}) = J_S(\overline{c})$.

Normalized welfare with a profit maximizing intermediary, W_B^P for buyers, W_S^P for sellers, and W_I^P for the intermediary, is

$$W_S^P = \int_0^{\overline{c}} (\overline{c} - c) dF_S(c) = \left(\frac{\beta}{\beta+1}\right)^{\beta+1} W_S^*, \qquad W_B^P = \int_{\underline{v}}^1 (v - \underline{v}) dF_B(v) = \left(\frac{\beta}{\beta+1}\right)^{\beta+1} W_B^*,$$

$$W_I^P = (\underline{v} - \overline{c}) F_S(\overline{c}) = \frac{\beta^\beta}{(\beta+1)^{\beta-1}} (W_S^* + W_B^*).$$

Note that both Walrasian and for-profit platform normalized welfare W_i^j with $i = S, B, I$ and $j = P, *$ are independent of δ.

No Intermediary Consider a setup in which there is no intermediary and buyers and sellers enter without having to pay any participation fees. For simplicity, also assume that they do not incur any exogenous participation costs either.

First, consider the case $\delta \to 0$. In this case, all sellers enter. This is because a seller with cost c close to 1 still has a positive probability of making the price offer. Since we are in an essentially static setup, a buyer with $v \geq c$ will accept the offer, which leads to a positive expected utility of sellers with $c < 1$. By the same reasoning, all buyers enter. Welfare for a seller with cost c is

$$W_S^N(c) = \alpha_S (P_S(c) - c)(1 - F_B(P_S(c))) + \alpha_B \int_0^1 \max\{0, P_B(v) - c\} dF_B(v),$$

where the price set by a seller is

$$P_S(c) = \arg\max_p (p - c)(1 - F_B(p)) = J_B^{-1}(c),$$

and by a buyer, $P_B(v) = J_S^{-1}(v)$. The analogous expression holds for the welfare $W_B^N(v)$ of a buyer with valuation v. Total (normalized) welfare for sellers $W_S^N = \int_0^1 W_S^N(c)dF_S(c)$ and buyers $W_B^N = \int_0^1 W_B^N(v)dF_B(v)$ can be computed as

$$W_S^N = \frac{\sqrt{\pi}\beta^\beta(\alpha_S+\beta)\Gamma(\beta+1)}{2^{2\beta+1}(\beta+1)^{\beta+1}\Gamma\left(\frac{3}{2}+\beta\right)}, \qquad W_B^N = \frac{\beta^\beta(\alpha_B+\beta)\Gamma(\beta+1)^2}{(\beta+1)^{\beta+1}\Gamma(2(\beta+1))},$$

where Γ is the gamma function.

When search frictions are small (the exit probability between two subsequent matches vanishes, $\delta \to 1$), it is known from the literature (see Satterthwaite and Shneyerov (2007) and Shneyerov and Wong (2010b)) that a dynamic random matching market without an intermediary converges to the Walrasian outcome (for any distribution). Therefore, for $\delta \to 1$ welfare is $W_B^N = W_B^*$ and $W_S^N = W_S^*$.

Welfare Comparison For small search frictions ($\delta \to 1$), buyer and seller welfare is clearly lower in the market with an intermediary ($W_B^P + W_S^P < W_B^N + W_S^N = W_B^* + W_S^*$). This is the standard result of a deadweight loss of monopoly, which comes from the monopolistic intermediary excluding buyers with $v \in [p^*, \underline{v}]$ and sellers with $c \in [\overline{c}, p^*]$ from trade.

However, for large search frictions ($\delta \to 0$), this result can be reversed. Numerical calculations reveal that for $\beta > 2$, welfare is higher with an intermediary than without ($W_B^P + W_S^P > W_B^N + W_S^N$).

The reason for this result is that in a search market, there may be excessive entry: inefficient buyers and sellers who would not trade in a Walrasian equilibrium ($v < p^*$ and $c > p^*$) enter, hoping to find a very efficient trading partner who prefers accepting an unattractive offer to incurring search costs to find another trading partner. A simple case illustrating this is a seller with $c = 1 - \epsilon$ in a static setup for some small positive ϵ. This seller is willing to enter, since she can make a positive expected profit by, for example, offering a price $1 - \epsilon/2$, which will be accepted with a (small) positive probability. The reason a buyer with $v \geq 1 - \epsilon/2$ accepts in a static setup is that he would get zero utility if he were to reject. Such a seller has a negative marginal contribution to welfare, since her offer is rejected with a probability close to 1, that is, $F_B(1 - \epsilon/2)$. In other words, most of the time she just holds up trade.[25]

An intermediary excludes some of the traders, which has the effect of reducing excessive entry additionally to the effect of the deadweight loss of monopoly. In some cases, the ensuing

[25] This is different from an auction, in which an additional inefficient entrant does not prevent the most efficient bidder from winning the auction.

efficiency gains dominate the standard deadweight loss of exclusion.[26]

6 Conclusions

We have derived an optimal dynamic mechanism for profit maximizing intermediaries. While the proof that the mechanism is optimal in the class of non-stationary, non-anonymous mechanism is rather lengthy, there is a very intuitive implementation of the mechanism. The implementation consists of charging stationary, anonymous participation fees and letting buyers and sellers bargain.

The equilibrium we consider is unique in the class of non-trivial balanced full-trade equilibria.[27] Outside of this class there may be additional equilibria. One can give the usual justification provided in the two-sided markets literature why one should focus on the profit-maximizing equilibrium: the platform can devise off-equilibrium payment schemes to destroy any other equilibrium than the profit-maximizing one. As an example, consider the trivial no-trade equilibrium, in which neither buyers nor sellers enter. The platform can offer subsidies to participants to "jump-start" the market in order to avoid this equilibrium.[28] Further, in a static setup, the profit-maximizing equilibrium can be shown to be the unique non-trivial equilibrium (see Appendix B).

This paper presents a novel analysis of dynamic for-profit intermediation and it could be extended in a number of directions. First, we have restricted attention to markets with a continuum of traders. This may be a useful simplification in the realm of e-commerce, whenever numerous traders participate in the platform. But obviously there are markets where this assumption may not be a good one. Deriving an optimal dynamic mechanism in this finite setting stands as an interesting open question. Second, while we have allowed for non-stationary mechanisms, the environment itself is stationary. It would be interesting to know how our results would generalize to non-stationary environments, when the distributions of types in the entering cohorts change over time. These extensions are left for future research.

[26]In the above example, overall welfare with an intermediary (including the intermediary's profits) is always higher in a static setup ($\delta \to 0$). However, one can easily construct examples in which the welfare comparison is ambiguous even in the static setup.

[27]For any other non-trivial balanced full-trade equilibrium, the marginal type conditions (13) and (14) would not be satisfied.

[28]This is e.g. the story of Diners Club, which initially gave vouchers to customers.

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Appendix

A Omitted Proofs

Proof of Proposition 1. A type-v buyer i who reports $\hat{v}$ to the mechanism will have expected utility

$$U_i^B(v, \hat{v}) = vQ_i^B(\hat{v}) - T_i^B(\hat{v}),$$

where

$$T_i^B(\hat{v}) := \sum_{s=0}^{\infty} \delta^{s-\tau(i)} t_{i,s}^B(\hat{v})$$

is the expected net present value of payments and

$$Q_i^B(\hat{v}) := \sum_{s=0}^{\infty} \delta^{s-\tau(i)} q_{i,s}^B(\hat{v})$$

is the ultimate discounted probability of trading (which can be viewed as the "expected net present value of the probability of trade"). Similarly, define $Q_j^S(c)$ and $T_j^S(c)$. Incentive compatibility implies the Envelope Formula

$$U_i^B(v,v) = \int_0^v Q_i^B(x)dx + U_i^B(0,0),$$

which can be used to back out the expected net present value of future payments

$$T_i^B(v) = vQ_i^B(v) - \int_0^v Q_i^B(x)dx - U_i^B(0,0).$$

A standard argument à la Myerson then implies that the expected payment made by type v buyer to the seller is equal to

$$T_i^B = \int_0^1 T_i^B(v)dF_B(v) - U_i^B(0,0) = \int_0^1 J_B(v)Q_i^B(v)dF_B(v) - U_i^B(0,0).$$

The expected time 0 present value of the intermediary's revenue from all buyers is equal to

$$T^B := \int_0^{\infty} \delta^{\tau(i)} T_i^B di = \int_0^{\infty}\int_0^1 \delta^{\tau(i)} J_B(v)Q_i^B(v)dF_B(v)di - \int_0^{\infty} \delta^{\tau(i)} U_i^B(0,0)di.$$

Similarly, the total monetary outlay to the sellers is

$$T^S := \int_0^{\infty} \delta^{\tau(j)} T_j^S dj = \int_0^{\infty}\int_0^1 \delta^{\tau(j)} J_S(c)Q_j^S(c)dF_S(c)dj + \int_0^{\infty} \delta^{\tau(i)} U_j^S(0,0)dj.$$

The intermediary's *objective* is to design a mechanism $\mathcal{M}$ that maximizes the expected profit

$$T = T^B - T^S \tag{33}$$

subject to the constraint (1) that trade is balanced in each period. As usual, the intermediary will choose to fully extract the surplus of the most inefficient types: $U_i^B(0,0) = U_j^S(1,1) = 0$.

Our key insight is that we can relax the constraint that trade balance has to hold in *each period*. Instead, we impose the weaker restriction that discounted trade balance has to be satisfied overall. Then we show that a solution of the relaxed problem exists that satisfies the stronger constraint.

Multiply the equation in (1) for each s by δ^s and add up:

$$\sum_{s=0}^{\infty} \delta^s \int_0^{\infty}\int_0^1 q_{i,s}^B(v)dF_B(v)di = \sum_{s=0}^{\infty} \delta^s \int_0^{\infty}\int_0^1 q_{j,s}^S(v)dF_S(c)dj.$$

Interchanging the order of summation and integration, and factoring out $\delta^{\tau(i)}, \delta^{\tau(j)}$, we get

$$\int_0^\infty \delta^{\tau(i)} \int_0^1 \sum_{s=0}^\infty \delta^{s-\tau(i)} q_{i,s}^B(v) dF_B(v) di = \int_0^\infty \delta^{\tau(j)} \int_0^1 \sum_{s=0}^\infty \delta^{s-\tau(j)} q_{j,s}^S(v) dF_S(c) dj.$$

The sums under the inner integrals can be recognized as discounted trading probabilities $Q_i^B(v), Q_j^S(c)$; substituting them, we obtain our *relaxed* trade balance constraint:

$$\int_0^\infty \int_0^1 \delta^{\tau(i)} Q_i^B(v) dF_B(v) di = \int_0^\infty \int_0^1 \delta^{\tau(j)} Q_j^S(c) dF_S(c) dj. \tag{34}$$

The intermediary's problem is to maximize profits as given by (33) subject to the constraint (34). Note that both (33) and (34) are linear in $Q_i^B(v), Q_j^S(c)$ and can be handled by a method similar to the one in Myerson and Satterthwaite (1983).

Specifically, similarly to the approach in Myerson and Satterthwaite (1983), we set up a Lagrangian

$$L := L^B + L^S,$$

where

$$\begin{aligned} L^B &:= \int_0^\infty \int_0^1 \delta^{\tau(i)} (J_B(v) - \lambda) Q_i^B(v) dF_B(v) di, \\ L^S &:= \int_0^\infty \int_0^1 \delta^{\tau(j)} (\lambda - J_S(c)) Q_j^S(c) dF_S(c) dj, \end{aligned}$$

and λ is the Lagrange multiplier.

Because L is linear in Q_i^B and Q_j^S, maximizing the Lagrangian with respect to $Q_i^B(v), Q_j^S(c)$ pointwise yields the bang-bang solution

$$Q_i^B(v) = \begin{cases} 1, & \text{if } J_B(v) \geq \lambda, \\ 0, & \text{otherwise,} \end{cases} \qquad Q_j^S(c) = \begin{cases} 1, & \text{if } J_S(c) \leq \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

The allocation rule provided in the proposition is obtained by choosing the marginal types $\underline{v}$, $\overline{c}$ such that

$$J_B(\underline{v}) = J_S(\overline{c}) = \lambda.$$

This optimal allocation can be implemented by setting *period* trading probabilities as in (3) in the proposition. This obviously satisfies the trade balance in each period provided the marginal types $\underline{v}, \overline{c}$ satisfy $1 - F_B(\underline{v}) = F_S(\overline{c})$. □

Proof of Proposition 3. Proof of part (i). Assume to the contrary that it is possible to implement the optimal (balanced full-trade equilibrium) mechanism with transaction fees. The intermediary charges the transaction fee κ_B to the buyer, κ_S to the seller. Buyers with valuation above $\underline{v}$ and sellers with costs below $\overline{c}$ enter. With probability α_B the buyer makes the offer, which is $\underline{p}_B = \overline{c} + \kappa_S$. With probability α_S the seller makes the offer, which is $\overline{p}_S = \underline{v} - \kappa_B$. The indifference condition for the marginal types in a full-trade equilibrium is

$$\alpha_B(\underline{v} - \underline{p}_B) = \alpha_B \kappa_B, \qquad \alpha_S(\overline{p}_S - \overline{c}) = \alpha_S \kappa_S,$$

by the same logic that leads to (15) and (16). The two equations are in fact one equation, $\kappa_B + \kappa_S = \underline{v} - \overline{c}$. It is useful to define the sum of fees as $\kappa := \kappa_B + \kappa_S$, since this will matter most for much of the following analysis. Note that $\kappa = \underline{v} - \overline{c}$ is a necessary, but not a sufficient condition for implementing the optimal mechanism as a full-trade equilibrium in a dynamic random matching market.

In the following, we will argue that the marginal seller $\overline{c}$ has an incentive to deviate by setting a price which will be rejected with positive probability. An analogous reasoning holds for the marginal buyer.

Consider the utility of a seller with cost c setting price p_S in a static setup ($\delta = 0$),

$$\pi_S(c, p_S) = \alpha_S(p_S - \kappa_S - c)(1 - \Phi_B(p_S + \kappa_B)\alpha_B(\underline{p}_B - \kappa_S - c),$$

which can be obtained by adding transaction fees to (22). It is useful to rewrite this in terms of the gross price $\hat{p}_S = p_S + \kappa_B$:

$$\begin{aligned}\pi_S(c, \hat{p}_S) &= \alpha_S(\hat{p}_S - (c + \kappa))(1 - \Phi_B(\hat{p}_S)) + \alpha_B(\underline{p}_B - \kappa_S - c) \\ &= \alpha_S \frac{\int_{\hat{p}_S}^{1} (J_B(v) - (c + \kappa)) dF_B(v)}{1 - F_B(\underline{v})} + \alpha_B(\underline{p}_B - \kappa_S - c),\end{aligned}$$

by the same logic as after (22). The derivative at $c = \overline{c}$, $\hat{p}_S = \underline{v}$ is positive,

$$\frac{\partial \pi_S(c, \hat{p}_S)}{\partial \hat{p}_S} = -\frac{\alpha_S(J_B(\hat{p}_S) - (c + \kappa)) f_B(\hat{p}_S)}{1 - F_B(\underline{v})} > 0,$$

since $J_B(\underline{v}) < \underline{v}$ and $\overline{c} + \kappa = \underline{v}$. Therefore, the marginal seller has an incentive to deviate from the full-trade equilibrium by setting a gross price $\hat{p}_S > \underline{v}$, which will be rejected with the positive probability $\Phi_B(\hat{p}_S) > 0$.

For a dynamic setup ($\delta > 0$), the same logic has to be applied as the one leading from (22) to (27) and the following expressions, that is, replacing Φ_B and J_B with their dynamic counterparts $\tilde{\Phi}_B$ and $\tilde{J}_B$. This still leads to a utility function by the seller which increases in $\hat{p}_S$ at $c = \overline{c}$, $\hat{p}_S = \underline{v}$, since

$$-(\tilde{J}_B(\underline{v}) - (\overline{c} + \kappa)) > 0, \tag{35}$$

and hence this cannot constitute a full-trade equilibrium.

Proof of part (ii). To achieve a full-trade equilibrium, the intermediary has to make sure that the marginal seller does not have an incentive to set a gross price higher than $\underline{v}$. That is, the inequality in (35) has to be reversed, which is equivalent to

$$\kappa \leq \tilde{J}_B(\underline{v}) - \overline{c}. \tag{36}$$

By an analogous reasoning, the marginal buyer will set a price $\overline{c} + \kappa_S$ if

$$\kappa \leq \underline{v} - \tilde{J}_S(\overline{c}). \tag{37}$$

Putting (36) and (37) together yields (29). This ensures that deviation from a full-trade equilibrium is not profitable.

The indifference condition for the marginal seller is

$$\frac{\min\{B,S\}}{S}\alpha_S(1-\Phi_B(\overline{p}_S+\kappa_B))(\overline{c}-\overline{p}_S) = K_S + \alpha_S\kappa_S,$$

which can be obtained by adding transaction fees to (14). In terms of the gross price $\hat{p}_S = \overline{p}_S + \kappa_B$, this is

$$\frac{\min\{B,S\}}{S}\alpha_S(1-\Phi_B(\hat{p}_S))(\overline{c}-\hat{p}_S-\kappa_B) = K_S + \alpha_S\kappa_S.$$

Because of $\min\{B,S\}/S = 1$ and $1-\Phi_B(\hat{p}_S) = 1$, this simplifies to

$$\alpha_S(\underline{v}-\overline{c}-\kappa) = K_S. \tag{38}$$

By the same logic for the marginal buyer, we get

$$\alpha_S(\underline{v}-\overline{c}-\kappa) = K_B. \tag{39}$$

Adding (38) and (39) yields the first part of (28); dividing (38) and (39) yields the second part. □